

RESEARCH STATEMENT

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1. INTRODUCTION

My research is in algebraic geometry and number theory. I am especially interested in abelian varieties and their moduli. My current work focuses on:

(a) *CM lifting (CML)*.

In characteristic 0, the theory of CM abelian varieties is of fundamental importance to number theory, and it is also a fascinating topic where analytic, algebro-geometric, and arithmetic techniques come together. On the other hand, in positive characteristic, by Honda-Tate theorem every abelian variety over a finite field has sufficiently many complex multiplication. The question (CML) asks whether every abelian variety over a finite field admits a CM lifting to characteristic 0. This question was first addressed by F. Oort in [13] (Thm. B), and a sharper version proved in [2] (3.5.6) says that (CML) does not hold in general.

In §2, I will elaborate the first part of my thesis, which studies the question of *strong CM lifting (sCML)*: let L be a CM field of degree $2g$, if we assume the whole ring of integers \mathcal{O}_L acts on a g -dimensional abelian variety A over a finite field \mathbb{F}_q , have we killed the obstructions to an L -linear CM lifting of A to characteristic 0? This question is a first step in studying which abelian variety over \mathbb{F}_q admits a CM lifting. I proved that the answer to question (sCML) is affirmative under the following additional assumptions on L : for every place v above p in the maximal totally real subfield L_0 of L , either v is inert in L , or v is split in L with absolute ramification index $e(v) < p - 1$. On the other hand, I also give several types of counterexamples to question (sCML). One of the counterexamples reveals an interesting phenomenon that concerns the reduction of the scheme-theoretic closure of finite subgroups in the geometric generic fiber of a CM abelian scheme in mixed characteristic. This is also one of the topics that I plan to study in the future.

(b) *The group action on the closed fiber of the universal deformation space of p -divisible groups*.

An important example of a p -divisible group is the inductive system of p -power torsion subgroups of an abelian variety over a field of characteristic p . By Serre-Tate theorem ([9] (1.2.1)), the infinitesimal deformation theory of abelian varieties over a field of characteristic p is equivalent to that of the attached p -divisible groups. For a p -divisible group X over $\overline{\mathbb{F}}_p$, the universal deformation space of X is a smooth formal scheme $\mathrm{Spf} \mathcal{R}$ over $W(\overline{\mathbb{F}}_p)$. There is a natural “relabelling action” on $\mathrm{Spf} \mathcal{R}$ by the compact p -adic group $G := \mathrm{Aut}(X)$. This group action and its equivariant cohomology of certain naturally defined vector bundle is deeply connected to stable homotopy theory.

One can get a good understanding of the action of G on the generic fiber of $\mathrm{Spf} \mathcal{R}$ via the p -adic period morphism developed by Dwork ([10]), Gross-Hopkins ([4]), and Rapoport-Zink ([14]). In the case when $\dim X = 1$ and the height of X is h , the p -adic period morphism is a G -equivariant étale surjective map

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between rigid analytic spaces: $\mathrm{Spf} \mathcal{R} \otimes_{W(\overline{\mathbb{F}}_p)} B(\overline{\mathbb{F}}_p) \rightarrow \mathbb{P}_{B(\overline{\mathbb{F}}_p)}^{h-1}$, where $\mathbb{P}_{B(\overline{\mathbb{F}}_p)}^{h-1}$ is the projective space of 1-dimensional quotients of the covariant isocrystal attached to X with a natural G -action. However, the p -adic period morphism does not provide much information about the action of G on the closed fiber of $\mathrm{Spf} \mathcal{R}$. By working with Cartier-Dieudonne theory, in [1] Chai obtained information on the leading terms of the action of G on $\mathrm{Spf} \mathcal{R}/p\mathcal{R}$ under an appropriate filtration on $\mathcal{R}/p\mathcal{R}$. Recently, Chai discovered a new approach via formal group laws to compute this action on the closed fiber of the Lubin-Tate space. This approach leads to an asymptotic formula of the action under appropriate filtration on $\mathcal{R}/p\mathcal{R}$. I am working on generalizing this approach in high-dimensional situations as the second part of my thesis; see §4 for the details.

2. CM LIFTING

Let L be a CM field of degree $2g$, L_0 be its maximal totally real subfield, and ι be the complex conjugation on L . We are interested in identifying the g -dimensional abelian varieties A equipped with $L \hookrightarrow \mathrm{End}^0(A)$ over a finite field \mathbb{F}_q , such that there exists a local domain R of characteristic 0 with residue field \mathbb{F}_q and an abelian scheme \mathcal{A} over R equipped with $L \hookrightarrow \mathrm{End}^0(\mathcal{A})$, satisfying $A_{\mathbb{F}_q}$ is L -linearly isomorphic to A . By deformation theory this question is geometric and local in nature, i.e., it can be reduced to the following question that concerns lifting subgroups of p -divisible groups over $\overline{\mathbb{F}}_p$:

Let F be a finite dimensional commutative \mathbb{Q}_p -algebra, Φ be a p -adic CM type for F , and F' be the reflex field. Let X be the unique O_F -linear CM p -divisible group over $R_0 := O_{F' \cdot B(\overline{\mathbb{F}}_p)}$ with p -adic CM type Φ . Let $X := X_{\overline{\mathbb{F}}_p}$ be the closed fiber. A subgroup G of X is said to be *liftable*, if there exists a finite extension R/R_0 and a finite locally free subgroup scheme \mathcal{G} of X_R , such that $\mathcal{G}_{\overline{\mathbb{F}}_p} = G$. What are the liftable groups of X ?

A complete list of liftable subgroups of X will allow us to identify all F -linear CM p -divisible groups over k that admit F -linear CM liftings with p -adic CM type Φ . The question (sCML) amounts to asking which O_F -stable subgroups of X are liftable, when we let $F = L \otimes_{L_0} L_{0,v}$ with v running over the places of L_0 above p , and consider all p -adic CM types Φ that are compatible the involution on F induced from ι .

In [7] and [8], I proved every O_F -stable subgroup is liftable for the following class of p -adic CM types (F, Φ) :

Theorem 2.1. *Suppose F/F^{ur} contains a subextension F_0/F^{ur} with $[F_0 : F^{ur}] < p - 1$. Let i_0 be an embedding of F^{ur} into $\overline{\mathbb{Q}}_p$. Let σ be the Frobenius automorphism on F^{ur} . For each integer k such that $0 \leq k \leq [F^{ur} : \mathbb{Q}_p]$, let \mathfrak{S}_k be the pullback of $\{i_0, i_0 \circ \sigma, \dots, i_0 \circ \sigma^{k-1}\}$ to $\mathrm{Hom}(F_0, \overline{\mathbb{Q}}_p)$. Let a be an integer such that $0 \leq a \leq [F^{ur} : \mathbb{Q}_p] - 1$, and Φ' be a subset of $\mathrm{Hom}(F_0, \overline{\mathbb{Q}}_p)$ such that $\mathfrak{S}_a \subset \Phi' \subset \mathfrak{S}_{a+1}$. Define Φ as the pullback of Φ' to $\mathrm{Hom}(F, \overline{\mathbb{Q}}_p)$. Let F' be the reflex field of (F, Φ) , and X be the O_F -linear CM p -divisible group over $R_0 := O_{F' \cdot B(\overline{\mathbb{F}}_p)}$ with p -adic CM type Φ .*

Then for every O_F -stable subgroup G of $X := X_{\overline{\mathbb{F}}_p}$, there exists a finite extension R over R_0 , such that G lifts to a finite locally free subgroup scheme of X_R .

As a corollary, I proved the following positive result concerning question (sCML):

Theorem 2.2. *If for every place v of L_0 above p , v is either inert in L , or split in L with absolute ramification index $e(v) < p - 1$, then for the CM field L the answer to question (sCML) for abelian varieties is affirmative.*

On the other hand, I also gave counterexamples to question (sCML). Let $F = L \otimes_{L_0} L_{0,v}$ and Φ be a p -adic CM type for F , then a liftable subgroup of X is defined over the residue field $\kappa_{F'}$ of the reflex field F' . This

causes an extra symmetry on the representation of \mathcal{O}_F on the Lie algebra of \mathcal{O}_F -linear CM p-divisible groups that admit F -linear CM liftings to characteristic 0. If there exists such a common extra symmetry when Φ runs over all p-adic CM types that are compatible with the involution on F induced from ι , then we obtain counterexamples to (sCML).

In the question on liftable subgroups of p-divisible groups, to give a complete list of liftable subgroups of $\mathcal{X}_{\overline{\mathbb{F}}_p}$, in general we need to let R run over all finite extensions of R_0 , and compute the reductions of all finite locally free subgroup schemes of \mathcal{X}_R . We had not seen any such attempts except for some very special cases, e.g., when \mathcal{X} or \mathcal{X}^\vee is Lubin-Tate.

I computed a first non-trivial example in [8]. In this example, $F = B(\mathbb{F}_{p^2})[\pi_0]/(\pi_0^2 - \epsilon p)$ where ϵ is a Teichmüller lift in $W(\mathbb{F}_{p^2})$ and is not a square, \mathcal{X} is a 2-dimensional \mathcal{O}_F -linear CM p-divisible group with a primitive p-adic CM type over $R_0 := W(\overline{\mathbb{F}}_p)[\pi_0]/(\pi_0^2 - \epsilon p)$. The answer is surprising: whether a finite subgroup of the geometric generic fiber of \mathcal{X} has an \mathcal{O}_F -stable reduction is completely determined by its order.

Theorem 2.3. *Suppose R is a finite extension of R_0 and \mathcal{G} is a finite locally free subgroup scheme of \mathcal{X}_R with order p^t , where t is an integer. Then we have the following description on the closed fiber $G := \mathcal{G}_k$ as a subgroup of $X := \mathcal{X}_{\overline{\mathbb{F}}_p}$:*

- (a) *If $t = 2n$ is even, then $G = X[\pi_0^n]$.*
- (b) *If $t = 2n + 1$ is odd, then $X[\pi_0^n] \subsetneq G \subsetneq X[\pi_0^{n+1}]$.*

In particular, the closed fiber G is \mathcal{O}_F -stable if and only if the order of \mathcal{G} is an even power of p .

Moreover, there is a description on the embedding of G between $X[\pi_0^n]$ and $X[\pi_0^{n+1}]$. For details, see [8] (2.2).

As a corollary, with the p-adic local field F as in the example above, I obtained a description on all F -linear CM p-divisible groups over $\overline{\mathbb{F}}_p$ that admit F -linear CM liftings with p-adic CM type compatible with the involution on F induced from ι . As a further corollary, I obtained a new type of counterexamples to question (sCML) that do *not* come from “small” residue fields of reflex fields. So the extra symmetry on the representation of \mathcal{O}_F on Lie algebra does not yet exhaust all the obstructions for an \mathcal{O}_F -linear CM p-divisible group to have an F -linear CM lifting.

3. THE GROUP ACTION ON THE CLOSED FIBER OF THE UNIVERSAL DEFORMATION SPACE OF P-DIVISIBLE GROUPS

Let Y be a connected p-divisible group over $\overline{\mathbb{F}}_p$, with $\dim Y = n$ and $\text{codim} Y = m$. The functor $\text{Def}_{W(\overline{\mathbb{F}}_p)}(Y)$ associates to an artinian local $W(\overline{\mathbb{F}}_p)$ -algebra R the set of isomorphism classes $[(\tilde{Y}, \alpha : \tilde{Y}_{\overline{\mathbb{F}}_p} \xrightarrow{\cong} Y)]$ of deformations of Y over R . This functor is represented by a smooth formal scheme $\text{Spf } \mathcal{R}$ of relative dimension mn over $W(\overline{\mathbb{F}}_p)$.

Let $G := \text{Aut}(Y)$. There exists a natural “relabelling action” of G on $\text{Def}_{W(\overline{\mathbb{F}}_p)}(Y)$. Namely, a group element $\rho \in G$ sends an isomorphism class of deformation $[(\tilde{Y}, \alpha)]$ to $[(\tilde{Y}, \rho \circ \alpha)]$. This induces an action of G on $\text{Spf } \mathcal{R}$. My advisor C.-L. Chai and I are interested in studying this action on the closed fiber $\text{Spf } \mathcal{R}/p\mathcal{R}$.

The main tool we employ is the theory of p-typical formal group laws. An (n -dimensional) formal group law $F = F(\underline{x}, \underline{y})$ over R is said to be *p-typical*, if $\mathbf{f}_l \gamma = 0$ for every formal curve $\gamma(t) \in R[[t]]^{n \times 1}$ and $l \neq p$, where \mathbf{f}_l is the l -th Frobenius operator on formal curves in F ; see [5] (15.7) or [19] (3.7) for details. When $\text{char} R = 0$, an equivalent definition for F to be p-typical is that the logarithm $\log_F(\underline{x})$ has the form of

$\sum_{i=0}^{\infty} a_i \underline{x}^{p^i}$, where a_i is an $n \times n$ matrix over R , and \underline{x}^{p^i} stands for the n -dimensional column vector with entries $x_1^{p^i}, x_2^{p^i}, \dots, x_n^{p^i}$. The functor that associates to R the set of (n -dimensional) p -typical formal group laws over R is represented by a polynomial ring $\tilde{\mathcal{R}}$ over \mathbb{Z} in infinitely many variables $V_i[j, k], 1 \leq j, k \leq n, i = 1, 2, \dots$; see [5] (10.3), (15.2.1), and (15.2.3).

Let \mathcal{Y} be the universal deformation of Y over \mathcal{R} , and let $\widehat{\mathcal{Y}}$ be the commutative smooth formal group attached to \mathcal{Y} . By Norman's algorithm in [12] and Cartier-Dieudonne theory developed by Cartier ([11], [19]) and Drinfel'd ([3]), a p -typical formal group law \mathcal{F} can be explicitly written down for $\widehat{\mathcal{Y}}$ under appropriate coordinates. In other words, there is an explicit embedding of the universal deformation space $\mathrm{Spf} \mathcal{R}$ into the formal completion $\mathrm{Spf} \tilde{\mathcal{R}}_x^\wedge$ of $\mathrm{Spf} \tilde{\mathcal{R}}$ at a certain closed point x such that \mathcal{F} is the pushforward of the universal p -typical formal group law $\tilde{\mathcal{F}}$ over $\tilde{\mathcal{R}}$.

The idea of computing the relabelling action of G on $\mathrm{Spf} \mathcal{R}$ is guided by the fact that the universal deformation space $\mathrm{Spf} \mathcal{R}$ is the "quotient" of $\mathrm{Spf} \tilde{\mathcal{R}}_x^\wedge$ modulo the equivalence that classifies \star -isomorphisms. Here a \star -isomorphism between formal group laws \tilde{F} and \tilde{G} over a commutative ring R equipped with $R \rightarrow \overline{\mathbb{F}}_p$ is an isomorphism whose pushforward over $\overline{\mathbb{F}}_p$ is identity. By Honda's method developed in [6], there is an explicit way to write down a "lift" of the relabelling action into an action of G on $\mathrm{Spf} \tilde{\mathcal{R}}_x^\wedge$. Namely, for each $g \in G$, there exists a homomorphism $\tilde{T}_g : \tilde{\mathcal{R}}_x^\wedge \rightarrow \tilde{\mathcal{R}}_x^\wedge$ such that the pushforwards of $\tilde{\mathcal{F}}$ and $\tilde{T}_{g*} \tilde{\mathcal{F}}$ to $\tilde{\mathcal{R}}$ are isomorphic, and the isomorphism between them reduces to g over the closed fiber. At the same time, there exists a unique homomorphism $\tilde{S}_g : \tilde{\mathcal{R}}_x^\wedge \rightarrow \tilde{\mathcal{R}}_x^\wedge$ such that $\tilde{S}_g \circ \tilde{T}_g$ stabilizes the $\mathrm{Spf} \mathcal{R}$ embedded in $\mathrm{Spf} \tilde{\mathcal{R}}_x^\wedge$, and $\tilde{T}_{g*} \tilde{\mathcal{F}}$ is \star -isomorphic to $(\tilde{S}_g \tilde{T}_g)_* \tilde{\mathcal{F}}$. The restriction of $\tilde{S}_g \circ \tilde{T}_g$ to $\mathrm{Spf} \mathcal{R}$ is the relabelling action of G .

In the Lubin-Tate case, an isomorphism between one-dimensional p -typical formal group laws decomposes into a strict isomorphism and rescaling of coordinates. C.-L. Chai made the key observation that with the help of universal strict isomorphisms between p -typical formal group laws, there exist integral recursive formulas for the actions \tilde{T}_g and \tilde{S}_g described above. This makes it possible to write down an asymptotic expansion formula of the relabelling action over the closed fiber.

As the second part of my thesis, I am in the progress of generalizing this approach to higher dimensional cases. With key features of C.-L. Chai's observation preserved, several new phenomena have arised. One difference from the one-dimensional case is that rescaling of coordinates does not necessarily preserve the p -typicality of a formal group law, therefore an extra p -typification procedure is needed.

4. FUTURE RESEARCH PLANS

4.1. Let F be a p -adic local field, Φ be a primitive p -adic CM type for F , and F' be the reflex field. Let \mathcal{X} be the \mathcal{O}_F -linear CM p -divisible group over $R_0 := \mathcal{O}_{F' \cdot B(\overline{\mathbb{F}}_p)}$. In the example in [8] about reductions of finite locally free subgroup schemes of \mathcal{X}_R where R runs over all finite extensions of R_0 , the computations indicate that the subgroup schemes seem to "try very hard" to have \mathcal{O}_F -stable reductions, though in characteristic 0 they may be far from being \mathcal{O}_F -stable. Based on this observation, the following question comes to me:

Problem 1. Is there a general condition on the p -adic CM type Φ , such that there exists an integer $d(\Phi)$ which only depends on Φ , satisfying that every liftable subgroup of $X := \mathcal{X}_{\overline{\mathbb{F}}_p}$ contains an \mathcal{O}_F -stable subgroup with index uniformly bounded by $p^{d(\Phi)}$? Recall that a subgroup G of X is said to be liftable, if there exists a finite extension R/R_0 and a finite locally free subgroup scheme $\mathcal{G} \subset \mathcal{X}_R$, such that $\mathcal{G}_{\overline{\mathbb{F}}_p} = G$.

In the case when $\#\Phi = 1$ or $[F : \mathbb{Q}_p] - 1$, this is true because every subgroup of X is O_F -stable, hence we could take $d(\Phi) = 0$. In the example I computed in [8], $\#\Phi = 2$ and $[F : \mathbb{Q}_p] = 4$, and the computations showed that we could take $d(\Phi) = 1$. Problem 1 will be one of my research plans in the future.

4.2. I am also considering the question of CM lifting with polarizations to normal domains up to an isogeny:

Problem 2. Let B be an abelian variety over \mathbb{F}_q with complex multiplication by a CM field L , and $\lambda : A \rightarrow A^\vee$ be an L -linear principal polarization. Does there exist a complete discrete valuation ring R of characteristic 0 with residue field \mathbb{F}_q , an abelian scheme \mathcal{A} over R with complex multiplication by L , and an L -linear principal polarization $\tilde{\rho} : \mathcal{A} \rightarrow \mathcal{A}^\vee$, such that there exists an L -linear quasi-isogeny φ between $\mathcal{A}_{\mathbb{F}_q}$ and B satisfying $\varphi^* \lambda = \tilde{\rho}_{\mathbb{F}_q}$? Here the L -structure on the dual abelian variety or scheme is defined to be the composition of the dual action with the complex conjugation.

This question is related to the reduction of special points on the Siegel moduli space.

4.3. Let X be a p -divisible group over $\overline{\mathbb{F}_p}$, and $\mathrm{Spf} \mathcal{R}$ be the universal deformation space of X . Concerned with the relabelling action of $G := \mathrm{Aut}(X)$ on the closed fiber of the universal deformation space, I am interested in the following questions:

Problem 3 What are the reduced irreducible formal subschemes of $\mathrm{Spf} \mathcal{R}/p\mathcal{R}$ that are stable under the relabelling action of an open subgroup of G ?

Problem 4 What happens if X is equipped with a quasi-polarization λ , and we consider the relabelling action of $\mathrm{Aut}(X, \lambda)$ on the closed fiber of the universal deformation space of (A, λ) ?

These questions are motivated by the aim of studying Hecke orbits on Shimura varieties of PEL types.

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